Lamb Shift.

23/2/21

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Hydrogen:

Schrö : $E \propto \frac{1}{n^2}$

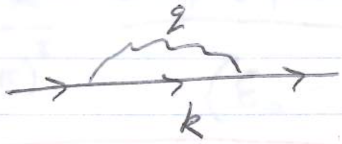
Dirac : $E = E(j)$

Exp : $2S_{1/2} - 2P_{1/2} \sim 1000$ MHz

$\sim 5 \cdot 10^{-6}$ eV

(Typical $E_{1s} \sim 13$ eV)

Self energy:



2 prop. $\frac{d^4k}{(2\pi)^4} \frac{\gamma^\chi k_\chi + m}{k^2 - m^2}$, $\frac{d^4q}{(2\pi)^4} \frac{g_{\mu\nu}}{q^2}$

too hard (for me)...

Semi-relativistic approach, H. Bethe Phys Rev 72, 339 (1947)

$$\mathcal{L}_{int} = -q A_\mu \bar{\psi} \gamma^\mu \psi$$

Coulomb gauge, $A_0 = 0$

$$\hat{h}_{int} = -q \gamma^0 \vec{\gamma} \cdot \vec{A} \rightarrow -q \vec{V} \cdot \vec{A}$$

non-rel electrons

$$\frac{p^2}{2m} \rightarrow \frac{1}{2m}(\vec{p} - q\vec{A})^2 = \frac{p^2}{2m} \underbrace{- q \frac{\vec{p} \cdot \vec{A}}{m}}$$

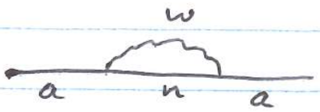
Regular QM : $\delta E_a = \sum_n \frac{\langle a | h | n \rangle \langle n | h | a \rangle}{\varepsilon_a - \varepsilon_n}$

Classically A = external field

QFT : $A \rightarrow \hat{A}$ photon operator

$$\hat{A} = \sum_{\varepsilon = \pm} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \hat{\varepsilon}\left(a_k^\dagger e^{ikx} + a_k e^{-ikx}\right)$$

$|a\rangle \to |a, 0\rangle$



$|n\rangle \to |n, 1\rangle$

$\varepsilon_n \to \varepsilon_n + \omega \quad (E_k = \omega)$

1 electron
1 photon

$$\boxed{\delta\varepsilon = \frac{e^2}{m^2} \sum_n \sum_\varepsilon \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{|\langle a, 0| \vec{p}\cdot\hat{\varepsilon}(a_k^\dagger e^{ikx} + a e^{-ikx})|n, 1\rangle|^2}{(\varepsilon_a - \varepsilon_n - \omega)\, 2\omega}}$$

Dipole approx: $kx \approx \omega t$

$$\boxed{\begin{array}{l} \lambda > a_B, \quad \frac{c\hbar}{\omega} > \frac{\hbar}{m_e c \alpha} \\ \omega < m_e c^2 \alpha \sim 10^4\,\mathrm{eV} \end{array}}$$

No $\vec{x}$-dep in photon part, $a^\dagger$ only acts on $|0\rangle, |1\rangle$

Photon part:

$$|\langle 0| \overbrace{a^\dagger e^{i\omega t} + a\, e^{-i\omega t}}^{y} |1\rangle|^2$$

$$= \langle 0|y|1\rangle\langle 1|y^\dagger|0\rangle$$

$$= \sum_m \langle 0|y|m\rangle\langle m|y^\dagger|0\rangle \qquad \text{trick: only } m=1 \text{ survives}$$

$$= \langle 0|yy^\dagger|0\rangle \qquad \text{closure}$$

only $aa^\dagger$ term in $yy^\dagger$ non-zero

$\underline{\langle 0|aa^\dagger|0\rangle = 1}$

$$\boxed{\begin{array}{l} a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \\ a|m\rangle = \sqrt{m}\,|m-1\rangle \end{array}}$$

electron part : $|\langle a|\vec{p}\cdot\hat{\varepsilon}|n\rangle|^2$

$$d^3\vec{k} = k^2\, d|k|\, d\Omega$$
$$= \omega^2\, d\omega\, d\Omega$$

Dipole result for spherically symmetric problem

$$\sum_{\varepsilon^\pm}\int |\langle \vec{p}\cdot\hat{\varepsilon}\rangle|^2 = \underset{\text{two } \varepsilon}{2}\ \underset{\substack{\text{no preffered direction} \\ \text{(just average)}}}{\left(\frac{4\pi}{3}\right)}\ |\langle\vec{p}\rangle|^2$$

$$\boxed{\delta\varepsilon = \underbrace{\frac{e^2}{m^2}\,\frac{4\pi}{3}\,\frac{1}{(2\pi)^3}}_{\frac{2\alpha}{3\pi}\frac{1}{m^2}}\sum_n |\langle a|\vec{p}|n\rangle|^2 \int_0^{\infty \to \omega_{max}} \frac{d\omega}{\varepsilon_a - \varepsilon_n - \omega}}$$

$$\frac{2\alpha}{3\pi}\frac{1}{m^2}, \quad \alpha = \frac{e^2}{4\pi} \approx 1/137 \quad (c=1,\ \hbar=1,\ \varepsilon_0=1)$$

- Linearly divergant
- Cutt-off ? a) V. sensitive
  b) Wrong answer

$$\omega_{max} \overset{?}{\approx} mc^2$$

Consider same correction to free e

- all same, except $|e_p\rangle$ mom. Eigen state

$\hookrightarrow \sum_{p'}$ " $\langle e_p|\vec{P}|e_{p'}\rangle = \langle e_p|\vec{P}|e_p\rangle$

only diag.

$$\delta E_{free} = \frac{e^2}{4\pi m^2}\frac{2}{3\pi}|\langle p\rangle|^2\int_0^{\omega_{max}}\frac{\omega}{-\omega}\,d\omega$$

$$= \frac{-2\alpha}{3\pi}\frac{|\langle\vec{P}\rangle|^2}{m^2}\omega_{max}$$

$\left(\alpha = \frac{e^2}{4\pi}\right)$ if $\begin{pmatrix}\hbar=1\\ c=1\\ \varepsilon_0=1\end{pmatrix}$

Also Linearly divergent.

$$= \cancel{}$$

$$= \frac{-4\alpha}{3\pi}\left(\frac{\omega_{max}}{m}\right)\frac{\langle p\rangle^2}{2m}$$

Correction to K·E $\Rightarrow$ $\frac{p^2}{2m_{obs}} = \frac{p^2}{2m_0} + C p^2$

$$\left(m \to m_{obs} \approx m_{bare}\left(1 + \frac{4\alpha}{3\pi}\frac{\omega_{max}}{m}\right)\right) \quad \left(\frac{1}{1+\varepsilon} \approx 1-\varepsilon\right)$$

Observed mass already contains this correction.

m in eq's actually m_{bare}

Renormalise:

$\hookrightarrow$ Correct for $\frac{p^2}{2m}$ in H!

$$\delta E^{obs} = \delta E - \langle a| \frac{\vec{p}^2}{2m_{obs}} - \frac{\vec{p}^2}{2m_0} |a\rangle$$

(Subtract off difference $\qquad H = \frac{\vec{p}^2}{2m} + V$

$$= \Delta E - \left[- \frac{2\alpha}{3\pi} \frac{\omega^{max}}{m^2} \right] \langle a|\vec{p}^2|a\rangle$$

$$\delta E = \frac{2\alpha}{3\pi} \frac{1}{m^2} \int_0^{\omega_{max}} \left(\sum_n \frac{|\langle a|\vec{p}|n\rangle|^2 \omega}{\varepsilon_a - \varepsilon_n - \omega} + |\langle a|\vec{p}^2|a\rangle \right) d\omega$$

$$= \qquad \langle a|p^2|a\rangle$$

$$= \sum_n \langle a|\vec{p}|n\rangle \langle n|\vec{p}|a\rangle$$

$$= \sum_n \frac{\varepsilon_a - \varepsilon_n - \omega}{\varepsilon_a - \varepsilon_n - \omega} |\langle a|\vec{p}|n\rangle|^2$$

$$\delta E = \frac{2\alpha}{3\pi} \frac{1}{m^2} \int_0^{\omega_{max}} \sum_n |\langle a|\vec{p}|n\rangle|^2 \left(\frac{\omega + \varepsilon_a - \varepsilon_n - \omega}{\varepsilon_a - \varepsilon_n - \omega} \right)$$

$$= \frac{2\alpha}{3\pi} \frac{1}{m^2} \sum_n |\vec{p}_{an}|^2 (\varepsilon_a - \varepsilon_n) \underbrace{\int_0^{\omega_{max}} \frac{1}{\varepsilon_a - \varepsilon_n - \omega} d\omega}$$

$$= \ln \left| \frac{|\varepsilon_n - \varepsilon_a|}{\omega^{max} - |\varepsilon_n - \varepsilon_a|} \right|$$

$$\approx - \ln \left(\frac{\omega_{max}}{E_a - E_n} \right)$$

Logarithmic divergence: OK.

$$\omega_{max} \sim m_e c^2 \sim 10^6 \text{ eV}$$

$$\mathcal{E}_a - \mathcal{E}_n \sim 10 \text{ eV} \quad \text{(typical excitation energy)}$$
$$= \Delta_{avg}$$

Large log: indep of n $\quad \ln\left(\frac{\omega_{max}}{\mathcal{E}_a - \mathcal{E}_n}\right) \simeq \ln\left(\frac{\omega_{max}}{\Delta_{avg}}\right)$

$$\delta E = \frac{2\alpha}{3\pi}\frac{1}{m^2} \underbrace{\sum_n \langle a|\vec{p}^{\,2}|n\rangle(\mathcal{E}_n - \mathcal{E}_a)} \ln\left(\frac{\omega_{max}}{\Delta_{avg}}\right) \quad \left(\begin{matrix}\text{swaped} \\ \mathcal{E}_n, \mathcal{E}_a, \\ \text{sign}\end{matrix}\right)$$

trick:

$$\sum_n \langle a|p^2|n\rangle(\mathcal{E}_n - \mathcal{E}_a) = \sum_n \langle a|\vec{p}|n\rangle \cdot \langle n|\vec{p}|a\rangle (\mathcal{E}_n - \mathcal{E}_a)$$

$$= \sum_n \langle a|\vec{p}|n\rangle \langle n|[H, \vec{p}]|a\rangle \qquad H = \frac{p^2}{2m} + V$$

$$= \langle a|\vec{p} \cdot [H, \vec{p}]|a\rangle$$

$$= \frac{1}{2}\langle a|[\vec{p}, [H, \vec{p}]]|a\rangle$$

$$[\vec{p}, [H, \vec{p}]] = (-i)^2 [\nabla, -\nabla V]$$

$$= \nabla^2 V \quad , \quad V = -\frac{Ze^2}{4\pi r} \qquad (\varepsilon_0 = 1)$$

$$= 4\pi Z\alpha\, \underline{\underline{\delta^3(\vec{r})}} \qquad = -\frac{Z\alpha}{r}$$

$$|\psi(0)|^2 = \frac{Z^3}{\pi a_0^3 n^3} = \frac{Z^3 m^3 \alpha^3}{\pi n^3} \quad \text{for s-states}$$

$\psi_0(0) = 0$ p-states

$(2s_{1/2} - 2p_{1/2})$

$$\delta \mathcal{E} = \frac{2\alpha}{3\pi}\frac{1}{m^2}\frac{4\pi Z\alpha}{2}\frac{(Z\alpha m)^3}{\pi n^3} \cdot \ln\left|\frac{\omega_{max}}{\Delta_{avg}}\right|$$

$$= \frac{8 Z^4 \alpha^3}{3\pi n^3} \cdot R_y \cdot \ln\left(\frac{mc^2}{\Delta_{avg}}\right) \quad , \quad R_y = \frac{m_e \alpha^2}{2} \sim 13\text{eV}$$

For $Z=1$, $n=2$ $\qquad \delta\mathcal{E} = \frac{\alpha^3}{3\pi} \cdot (13\text{eV}) \cdot 10 \simeq 5 \times 10^{-6} \text{ eV} \,!$