B

# Quantise Dirac + QED

1/3/21

Ben Roberts

cf Scalar: $\mathcal{L} \sim \partial_\mu \phi \partial^\mu \phi$

$$\hat{\phi} = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left( \hat{a}_k e^{-ikx} + a^\dagger_k e^{ikx} \right)$$

$$[a_k, a^\dagger_{k'}] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$$

---

Dirac $\quad \mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi \quad , \quad \pi = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = i\Psi^\dagger$

$$\hat{\Psi}(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[ \hat{b}^s_{\vec{p}} u^s(p) e^{-ipx} + \hat{c}^{\dagger s}_{\vec{p}} v^s(p) e^{ipx} \right]$$

$\bar{\Psi}$ : Same, $\dagger, \gamma^0 \dagger$

---

Photon

$$A_\mu(x) = \sum_s \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \varepsilon^s_\mu \left[ a^s_k e^{-ikx} + a^{\dagger s}_k e^{ikx} \right]$$

$\partial^2_\mu A^\mu = 0$ : Lorentz gauge.

$\varepsilon$: only 2 indep pol vectors

$\varepsilon^s \cdot \vec{k} = 0 \qquad$ e.g. $k = k_z$, $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\hat{x}, \hat{y}$ ...

---

$$\{b^s_{\vec{p}}, b^{\dagger r}_{\vec{q}}\} = \{c^s_{\vec{p}}, c^{r\dagger}_{\vec{q}}\} = (2\pi)^3 \delta^{rs} \delta(\vec{p} - \vec{q})$$

$\{ \ \}$ rest $= 0$

---

$$[a^s_{\vec{k}}, a^{\dagger r}_q] = (2\pi)^3 \delta^{rs} \delta(\vec{p} - \vec{q})$$

Fermions, Bosons

Fermions $\quad |p\rangle \propto b^\dagger_p |0\rangle$

$$|p,q\rangle \propto b^\dagger_q b^\dagger_p |0\rangle$$

$$|q,p\rangle \propto b^\dagger_p b^\dagger_q |0\rangle$$

$$= -|p,q\rangle$$

Since $\{ b^\dagger_p, b^\dagger_q \} = 0$

→ Fermion anti-symmetry ; hence anti-commutator

---

1- particle states: $\quad |\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}}\, a^{\dagger s}_{\vec{p}} |0\rangle$

---

$$H_{Dirac} = \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left( b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} + c^{s\dagger}_{\vec{p}} c^s_{\vec{p}} \right)$$

$$\mathcal{L}_{QED} = i \bar{\psi} \gamma^\mu (\partial_\mu + i q_e A_\mu) \psi - m \bar{\psi}\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$= i \bar{\psi} (\not{D} - m) \psi - \frac{1}{4} F^2$$

$$= \mathcal{L}_{Dirac}(\psi) + \mathcal{L}_\gamma(A) + \mathcal{L}_{int}(\psi, A)$$

$$\boxed{\mathcal{L}_{int} = - q A_\mu \bar{\psi} \gamma^\mu \psi}$$

# Perturbation Theory

S-matrix

$$|\psi(t)\rangle = \underbrace{\hat{U}(t,t_0)}\,|\psi(t_0)\rangle$$

$$U = \exp\left(-i\int_{t_0}^{t} H_{int}(t')\,dt'\right)$$

$$\simeq 1 - i\int H\,dt + \tfrac{1}{2}(-i)^2 \iint H\cdot H\,dt\,dt + \dots$$

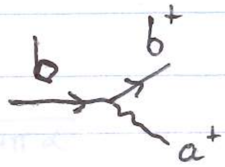
~~First-order~~ $$S = \langle f|U|i\rangle\Big|\ \lim\ \begin{matrix} t_0 \to -\infty \\ t \to +\infty \end{matrix}$$

First-order : $$H_{int} \sim \int A_\mu \bar{\psi}\gamma^\mu\psi$$

$$A_\mu \sim a + a^\dagger$$
$$\psi \sim b + c^\dagger$$
$$\bar{\psi} \sim c + b^\dagger$$

First-order : combos of ↑

$a^\dagger b^\dagger b$




$c^\dagger b^\dagger a$


+ 8 ~~others~~ total

$2^{nd}$ order e.g.

↳ Since $a|0\rangle = 0$, must have $aa^\dagger$ pair to survive

↳ eval'd diff $\propto$

↳ propogator

↳ "Motivation", not proof, of F.Rs

# Feynman Rules for QED

 $u^s$ in $e^-$ |  $\bar{u}^s$ out $e^-$

 $\bar{v}^s$ in $e^+$ |  $v^s$ out $e^+$

 $\varepsilon^{s,\mu}$ in $\gamma$ |  $\varepsilon^{s\mu *}$ out $\gamma$

$e^-$ propogator  $i \dfrac{d^4 q}{(2\pi)^4} \dfrac{\gamma^\mu q_\mu + m}{q^2 - m^2}$

$\gamma$ prop.  $(-i) \dfrac{d^4 q}{(2\pi)^4} \dfrac{g_{\mu\nu}}{q^2}$

Vertex : $-i\, q_e \gamma^\mu (2\pi)^4 \delta(\Delta p)$ in-out

$$q_e = +|e| = \sqrt{4\pi\alpha}$$

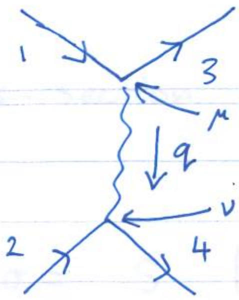
- Include relative sign between D's w/ exchange
  $\{b^+, b^+\} = 0$

Diagram : $-i (2\pi)^4 \delta(\Delta p) \underline{\underline{M}}$

$$M = \int \langle f | H_{int}^{(t)} | i \rangle dt.$$

$u_{(1)} \equiv u^{s_1}(p_1)$

E.g.




Time $\longrightarrow$

$$\bar{u}(3)\,(-i g_e)\,\gamma^\mu\, u(1) \quad (2\pi)^4\,\delta(p_1 - p_3 - q)$$

$$\times \quad -i \,\frac{d^4 q}{(2\pi)^4}\, \frac{g_{\mu\nu}}{q^2}$$

$$\times \; \bar{u}(4)\,(-i g_e)\,\gamma^\nu\, u(2) \quad (2\pi)^4\,\delta(p_2 - p_4 + q)$$

$$\int d^4 q \quad \rightarrow \quad q = p_1 - p_3$$

$$\Rightarrow \qquad i \underbrace{\boxed{e^2 \;(\bar{u}_3 \gamma^\mu u_1)\, \frac{g_{\mu\nu}}{(p_1 - p_3)^2}\, \bar{u}_4 \gamma^\nu u_2}}_{= -M} \quad (2\pi)^4\,\delta(p_2 - p_4 + p_1 - p_3)$$

---

Meaning: Non-rel limit

$$\bar{u}\gamma^\mu u = \bar{u}\gamma^0 u \quad \text{for } \vec{p} = 0$$

$q = p_1 - p_3$ elastic, non-rel

$$q^2 = -|\vec{q}|^2$$

$$M \simeq \frac{e^2\,(\bar{u}\gamma^0 u)\,(\bar{u}\gamma^0 u)}{-|\vec{q}|^2}$$

$$= \frac{-e^2 \,(u^\dagger u)\,(u^\dagger u)}{|\vec{q}|^2} \qquad u^\dagger u = 2m\,\delta_{ss'}$$

$V(\vec{q})$

$$V(r) = e^2 \int \frac{d^3 q}{(2\pi)^3}\, \frac{1}{|\vec{q}|^2}\, e^{i\vec{q}\cdot\vec{r}}$$

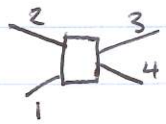
$$= \frac{e^2}{4\pi r} = \frac{\alpha}{r}$$

## Cross - section

$$d\sigma = S \quad |\mathcal{M}|^2 \quad (2\pi)^4 \quad \delta(\Delta p) \; \frac{1}{4 E_1 E_2 v} \prod^{\text{out}} \left( \frac{d^3 p_o}{(2\pi)^3 \, 2 E_o} \right)$$

$$v = |\vec{v}_1 - \vec{v}_2|$$

Typical : 2-particles in final state. (3, 4)




$$\int d^3 p_3 \underbrace{\int d^3 p_4}_{} \qquad d|p| \, p^2 \, d\Omega$$

$$E_{cm} = E_1 + E_2$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{|\vec{p}_4|}{4 E_1 E_2 \; |v_1 - v_2|} \; \frac{1}{(2\pi)^2 \; 4 \; E_{cm}} \; |\mathcal{M}|^2$$

- Typical : not spin - polarised

  → Sum over final-state spins

  → Average over initial

$$\to \left( \frac{1}{4} \sum_{s_1} \sum_{s_2} \right) \sum_{s_3, s_4 \ldots} |\mathcal{M}|^2$$

- To evaluate, make use of completeness

$$\sum_s u_p^s \bar{u}_p^s = \gamma^\mu p_\mu + m$$

$$\sum_s v_p^s \bar{v}_p^s = \gamma^\mu p_\mu - m$$

- Be careful re-ordering $\gamma$'s

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

- $|M|^2 = M^\dagger M$

Eg: $\sum_{s,r} (\bar{u}_3^s \gamma^\mu u_1^r)(\bar{u}_4^r \gamma^\nu u_2^s)$

$$= \sum_{s,r} (\bar{u}^s(p) \gamma^\mu u^r(q))(\bar{u}^r(q) \gamma^\nu u^s(p))$$

$$= \sum_{sr} (\bar{u}_a^s \gamma_{ab}^\mu u_b^r) \cdot (\bar{u}_c^r \gamma_{cd}^\nu u_d^s) \qquad \text{4x4 Matrix mult. Explicit}$$

$$= \left(\sum_s \bar{u}_a^s(p) u_d^s(p)\right)\left(\sum_r u_b^r(q) \bar{u}_c^r(q)\right) \gamma_{ab}^\mu \gamma_{cd}^\nu$$

$$= (\not{p} + m)_{da} \underbrace{\gamma_{ab}^\mu (\not{p} + m)_{bc}}_{M_{bc}} \gamma_{cd}^\nu$$

$$= \mathrm{Tr}\left((\not{p} + m)\gamma^\mu (\not{p} + m)\gamma^\nu\right)$$

$$\mathrm{Tr}(\gamma^\mu) = 0$$

$$\mathrm{Tr}(\text{odd}\ \#\ \gamma^\mu) = 0$$

Proof

$$\mathrm{tr}(\gamma^\mu) = \mathrm{tr}(\gamma^5\gamma^5\gamma^\mu) \qquad (\gamma^5)^2 = 1$$

$$= -\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^5) \qquad \{\gamma^\mu, \gamma^5\} = 0$$

$$= -\mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu)$$

$$(\mathrm{Tr}(A \cdot B \cdot C) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB))$$

$$\Rightarrow = 0$$

$$\mathrm{Tr}(1) = 4$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4 g^{\mu\nu}$$