Dirac: Intro/refresh

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Free-field Dirac equation

$$(\gamma^\mu p_\mu - m)\psi = 0$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$(i\not{\partial} - m)\psi = 0 \qquad \not{a} = \gamma^\mu a_\mu$$

$$(\gamma^\mu p_\mu + m)(\gamma^\nu p_\nu - m)\psi = 0$$

$$= (\varepsilon^2 - \vec{p}^2 - m^2)\psi = 0 \quad ? \quad \text{(K-G equation)}$$

$$\Rightarrow \quad (\gamma^0)^2 = 1 \,, \quad (\gamma^a)^2 = -1 \qquad a = 1,2,3$$

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \qquad \text{for } \mu \neq \nu$$

$$\Rightarrow \quad \boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}} \qquad \left(g^{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \text{ e.g.}\right)$$

Simplest: 4x4

Chiral (Weyl) rep: $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}$

- Good for high energy / QED

Dirac rep: $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}$

- Good for atomic: nice non-rel. limit

σ^a Pauli spin, e.g. $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Also $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \,, \quad (\gamma^5)^2 = 1 \,, \quad \{\gamma^5, \gamma^\mu\} = 0$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$ Dirac "bi"-Spinor

$\psi^\dagger \psi$ - not Lorentz Invariant

$\psi^\dagger \gamma^0 \psi$ is $\Rightarrow$ $\boxed{\bar{\psi} \equiv \psi^\dagger \gamma^0}$

Free-field lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = \frac{\partial \mathcal{L}}{\partial \phi} \qquad \phi = \begin{cases} \psi \\ \bar{\psi} \end{cases}$$ $\leftarrow$ D.E for e

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q=|e|
$c = \varepsilon_0 = \hbar = 1$

$\psi \to \psi e^{+iq\theta(x)}$: must introduce A to make $\mathcal{L}$ inver.

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu + iqA_\mu)\psi - m\bar{\psi}\psi$$

$$A_\mu \to A_\mu + \partial_\mu \theta$$ (cancels left-over derivative from $\partial_\mu(\psi e^{-iq\theta})$

• only anti-symmetric combo's of ∂A leav $\mathcal{L}$ invariant. factor: by convention

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu(\partial_\mu + iqA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\boxed{\mathcal{L}_D = i\bar{\psi}(\not{D} - m)\psi - \frac{1}{4}F^2}$$

F ? $$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \qquad ①$$

↳ just from form of anti-symm F

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -q\bar{\psi}\gamma^\mu\psi$$

$$\partial_\mu F^{\mu\nu} = \underbrace{q\bar{\psi}\gamma^\mu\psi}_{J^\mu} \qquad ②$$

e.g. $\nu = 0$ in ② $\quad \nabla \cdot \vec{E} = \underset{\sim \rho}{J^0}$

if $\left(\begin{array}{l} A^\mu = (A_0, \vec{A}) \\ \vec{E} \equiv -\nabla A_0 - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} \equiv \nabla \times \vec{A} \end{array} \right)$

$\{\lambda, \mu, \nu\}$ in ① $\Rightarrow \{0,1,2\} \rightarrow \nabla \times E = -\frac{\partial B}{\partial t}$

① + ② ⇒ Maxwells Eqs.

Free - Dirac Solutions

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0$$

Rest - frame $\quad p = (m, \vec{0})$

$$(i\gamma^0 \partial_0 - m)\,\psi = 0$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Block γ-matrix

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$i\partial_0 \psi_L - m\psi_R = 0$$

$$i\partial_0 \psi_R - m\psi_L = 0$$

4 indep solutions: $\quad u^\pm,\ v^\pm$

$$\underset{u^+}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-imt}}, \quad \underset{u^-}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} e^{-imt}}, \quad \underset{v^+}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} e^{imt}}, \quad \underset{v^-}{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{imt}}$$

spin $\times 2$, $\quad e^-/e^+$

$$(i\vec{\gamma}\cdot\nabla + m)\psi = -i\gamma^0 \partial_t \psi \qquad (\gamma^0)^2 = 1$$

$$\underbrace{(i\gamma^0\vec{\gamma}\cdot\nabla + \gamma^0 m)}_{H_D}\,\psi = -i\partial_t \psi$$

$H_D u = m u \quad \rightarrow$ +ve energy $\quad$ electrons

$H_D v = -m v \quad \rightarrow$ "-ve energy" $\quad$ positrons

} "-ve energy" $\rightarrow$ opposite charge +ve energy when QED.

Plane-Wave: Boost

e.g. $$u^{+}(p, x) = C_{Norm} \begin{pmatrix} 1 \\ 0 \\ (E_p + P_z)/m \\ 0 \end{pmatrix} e^{-ip\cdot x}$$

for $\vec{p} = P_z \hat{z}$

$$u^s(p,x) = u^s(p)\, e^{-ip\cdot x}$$

Normalisation

$$\bar{u}^r(p)\, u^s(p) = 2m\, \delta_{rs}$$

$$\bar{V}^r(p)\, V^s(p) = -2m\, \delta_{rs}$$

$\bar{u}\cdot V = 0$ etc.

Complete (in s-space) as:

$$\sum_{s=\pm} u^s(p)\, \bar{u}^s(p) = \gamma\cdot p + m$$

$$\sum_{s=\pm} V^s(p)\, \bar{V}^s(p) = \gamma\cdot p - m$$

(not proved here)

$$u^s(p) = \begin{pmatrix} \sqrt{E - \sigma\cdot\vec{p}}\; \chi^s \\ \sqrt{E + \sigma\cdot\vec{p}}\; \chi^s \end{pmatrix}, \quad V^s(p) = \begin{pmatrix} '' \\ -'' \end{pmatrix}$$

$\chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ [e.g]

↳ eigenstates of σ_z